THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050A Midterm solution

1. (a) Let $\epsilon > 0$. Take $N \in \mathbb{N}$ such that $N > 3/\epsilon$.
   If $n > N$,

$$\begin{aligned} \left|\frac{n^2-1}{n^2+n+1} - 1\right| &= \left|\frac{n+2}{n^2+n+1}\right| \\ &\leq \left|\frac{n}{n^2+n+1}\right| + \left|\frac{2}{n^2+n+1}\right| \\ &< \left|\frac{n}{n^2}\right| + \left|\frac{2}{n}\right| \\ &= \frac{3}{n} \\ &\leq \frac{3}{N} \\ &< \epsilon \end{aligned}$$

   (b) Since $(n+1)^2 = n^2 + 2n + 1 \geq 4n$, for all $n \in \mathbb{N}$,

$$\frac{\sqrt{n}}{n+1} \leq \frac{1}{2}$$

   Pick $\epsilon_0 = \frac{1}{2}$. Then for any $n \in \mathbb{N}$,

$$\begin{aligned} \left|\frac{\sqrt{n}}{n+1} - 1\right| &= 1 - \frac{\sqrt{n}}{n+1} \\ &\geq \frac{1}{2} \\ &\geq \epsilon_0 \end{aligned}$$

2. (nx_n) is convergent sequence, then there exists $l \in \mathbb{R}$ such that

$$\lim nx_n = l$$

Let $\epsilon_0 = 1$, there exist $N_1 \in \mathbb{N}$ such that if $n \geq N$,

$$\begin{gathered}|nx_n - l| < \epsilon_0 \\ l - 1 < nx_n < l + 1 \\ \frac{l-1}{n} < x_n < \frac{l+1}{n}\end{gathered}$$

Take $C = \max\{|l = 1|, |l - 1|\}$

$$|x_n| < \frac{C}{n}$$

Let $\epsilon > 0$. Choose $N_2 \in \mathbb{N}$ such that $N_2 > \epsilon/C$
Then, take $N = \max\{N_1, N_2\}$. If $n \geq N$

$$|x_n - 0| < \frac{C}{n} < \frac{C}{N_2} < \epsilon$$

3. (a) Let (x_n) be a Cauchy sequence and let $\epsilon = 1$.
   There exist $H \in \mathbb{N}$ such that if $n \geq H$, then $|x_n - x_H| < 1$.
   Hence, we have $|x_n| \leq |x_H| + 1$ for all $n \geq H$.
   Then, set $M = \sup\{|x_1|, |x_2|, \cdots, |x_{H-1}|, |x_H| + 1\}$,
   Thus, $|x_n| \leq M$ for all $n \in \mathbb{N}$.

   (b) For any $N \in \mathbb{N}$, choose odd number $n > N$. Let $m = n + 1$, m is even number.
   Take $\epsilon_0 = 1$.

$$\begin{aligned}|x_m - x_n| &= \left|\frac{2m+m}{3} - \frac{2n-n}{3}\right| \\ &= \left|\frac{2 + (2n+1)}{3}\right| \\ &= \frac{3+2n}{3} \\ &\geq 1 \\ &= \epsilon_0\end{aligned}$$

4. Let $\epsilon > 0$.
   Since $\lim r^n = 0$, there exist $N \in \mathbb{N}$ such that $|r^n| < \epsilon(1 - r)$ for all $n > N$.
   Take this N, if $m > n \geq N$,

$$\begin{aligned}|y_m - y_n| &\leq |y_m - y_{m-1}| + |y_{m-1} - y_{m-2}| + \cdots + |y_{n+1} - y_n| \\ &< r^n + r^{n-1} + \cdots + r^{m-2} + r^{m-1} \\ &< r^n + r^{n+1} + \cdots \\ &= \frac{r^n}{1-r} \\ &< \epsilon\end{aligned}$$

   Then, (y_n) is Cauchy sequence.
   Thus, by Cauchy Convergence Criterion, (y_n) is convergent and $\lim y_n$ exists.